UDC 004.4

Interpolation using stochastic local iterated function systems

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Abstract. We will use iterated function systems to construct fractal functions. Local iterated function systems are important generalization of iterated function systems. The classical data interpolation methods can be generalized with fractal functions. In this article using the fact that graphs of piecewise polynomial functions can be written as the fixed points of local iterated function system we study the behavior of stochastic local iterated function systems using interpolation methods.

Keywords: fractal function, iterated function system, local iterated function system, stochastic local iterated function system .

1. Introduction

A function $f : I \to \mathbf{R}$, defined on the real closed interval I is named by Barnsley a *fractal function* if the Hausdorff dimension of the graph is noninteger. Barnsley introduced in [1] the notion of fractal interpolation function (FIF). He said that a fractal function is a (FIF) if it possess some interpolation properties. In the last few decades the methods of fractal interpolation methods was applied successfully in many fields of applied sciences. It has the advantage that it can be also combine with the classical methods or real data interpolation. Hutchinson and Rüschendorf [6] gave the stochastic version of fractal interpolation function. In order to obtain fractal interpolation functions with more flexibility Wang and Yu [9] use instead of a constant scaling parameter a variable vertical scaling factor. Barnsley in [?] and [3] introduced the notion of local iterated function systems which are an important generalisation of the global iterated function systems. In this paper we study the case of a locally stochastic fractal interpolation function with random variable as scaling parameter.

2. Local iterated function systems and local fractal functions

The notion of local iterated function systems was introduced in [4] and is a generalization of a classical or general iterated function system IFS. Let (X, d_X) be a omplete metric spaces with metric d_X and $N = \{1, 2, 3, ...\}$ the set of positive integers.

Let $n \in N$ and $N_n = \{1, 2, ..., n\}$, and consider a family of nonempty subsets of X, $\{X_i | i \in N_n\}$. Assume that there exists continuous mapping

$f_i : X_i \to X, i \in N_n$, for each X_i. Then $\mathbb{F}_{loc} = \{X, (X_i, f_i)|i \in N_n\}$ is called a **local iterated function system.**

If each $X_i = X$ then this definition give us the usual definition of a global iterated function system on a complete metric space.

A mapping $f : Y \subset X \to X$ is a contraction on Y if exists a constant $\lambda \in [0, 1)$ such that

$$d_X(f(x_1), f(x_2)) \le \lambda d_X(x_1, x_2), \quad \forall x_1, x_2 \in X.$$

A local IFS $\mathbb{F}_{loc}$ is called contractive if there exists a metric d' equivalent to d_X with respect to which all functions $f \in \mathbb{F}_{loc}$ are contractive, on their respective domains.

Let be the power set of X, $2^X = \{S|S \subset X\}$. On this set we consider a set-valued operator using a local IFS:

$$\mathbb{F}_{loc}(S) = \cup_{i=1}^{n} f_i(S \cap X_i), \tag{1}$$

where $f_i(S \cap X_i) = \{f_i(x)|x \in S \cap X_i\}$.

A subset $G \in 2^X$ is called a local attractor for the local IFS $\{X, (X_i, f_i)|i \in N_n\}$ if

$$G = \mathbb{F}_{loc}(G) = \cup_{i=1}^{n} f_i(G \cap X_i).$$

For example the empty set is a local attractor of the local IFS, and if G_1 and G_2 are distinct local attractors than $G_1 \cap G_2$ is also a local attractor. So, exists a largest local attractor for the IFS, and this will be the so-called local attractor of the local IFS. In the case when X is compact and $X_i, i \in N_n$ are also compact in X, and the local IFS $\{X, (X_i, f_i)|i \in N_n\}$ is contractive, the local attractor may be computed in the following way. Let $L_0 = X$ and

$$L_n = \mathbb{F}_{loc}(L_{n-1}) = \cup_{i \in N_n} f_i(L_{n-1} \cap X_i), \quad n \in \mathbb{N}.$$

Then $\{L_n|n \in N_0\}$ is a decreasing nested sequence of compact sets. If each L_n is nonempty, then by the Cantor intersection theorem,

$$L = \cap_{n \in N_0} L_n \neq \emptyset,$$

we have that

$$L = \lim_{n \to \infty} L_n,$$

where the limit is taken with respect to the Hausdorff metric. This implies that

$$L = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \cup_{i \in N_n} f_i(L_{n-1} \cap X_i) = \cup_{i \in N_n} f_i(L \cap X_i) = \mathbb{F}_{loc}(L).$$

Follows that, $L = G_{loc}$. Also Barnsley introduce the local fractal functions as the local attractors which are the graphs of bounded

functions.
Let X **be a nonempty connected set and** $\{X_i | i \in N_n\}$ **are subsets of** X **which are nonempty and connected. We will consider a family of bijective mappings,** $v_i : X_i \to X, i \in N_n$ **such that** $\{v_i(X_i), i \in N_n\}$ **is a kind of partition of** X**,** $X = \cup_{i=1}^n v_i(X_i)$ **and** $v_i(X_i) \cap v_j(X_j) = \emptyset, \quad \forall i \neq j \in N_n$**.**
Let be also (Y, d_y) **a complete metric space with the metric** d_y**, than a function** $f : X \to Y$ **is called is bounded with respect to the metric** d_y**, if exists** $M > 0$ **such that** $\forall x_1, x_2 \in X, d_Y(f(x_1), f(x_2)) < M$**.**
Than the space $D(X,Y) = \{f : X \to Y | f \ \textit{is bounded}\}$**, with the metric** $d(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$ **is a complete metric space,** $(D(X,Y), d)$**. In a similar way we can define** $D(X_i, Y)$**, for all** $i \in N_n$ **and let be** $f_i = f|X_i$**. We will consider now a set of functions which are uniformly contractive in the second variable** $w_i : X_i \to X, \ \ i \in N_n$**, and the Read-Bajactarević operator** $B : D(X,Y) \to Y^X$ **defined by**

$$Bf(x) = \sum_{i=1}^{N} w_i(v_i^{-1}(x), f_i \circ v_i^{-1}(x)) \chi_{u_i(X_i)}(x),$$

where

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

Using the contraction property's on the second variable of the applications w_i**, it follows that the operator** B **is also a contraction on the complete metric space** $D(X,Y)$ **and therefore it has a unique fixed point** f^* **in** $D(X,Y)$**. This unique fixed point will be called a local fractal function, generated by** B**.**

Next we consider the random version of the above construction.

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