UDC 519.21

Harmonic Analysis of Random Walks on Lattices

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Abstract. We consider continuous-time symmetric, spatially homogeneous, irreducible branching random walks on the multidimensional lattice. Corresponding transition intensities of the random walk have heavy tails. This assumption implies that the variance of jumps becomes infinite. The growth rate estimate of the Fourier transform for transition intensities of random walk on $\mathbb{Z}^3$ is obtained.

Keywords: branching random walks, Fourier transform, Watson lemma.

1. Introduction

We consider a symmetric branching random walk (BRW) on $\mathbb{Z}^d$ with continuous time and only branching source at the origin. Assume that initially there is a single particle located at site $x \in \mathbb{Z}^d$ and it jumps on the lattice until walking into the origin. Here particle dies and starts a branching process, giving birth to offsprings. Then each of these new particles walk independently according to the same law. The main objects of interest are the local, i.e. at site of lattice, and total numbers of particles at an arbitrary moment of time.

This BRW model for a simple random walk with one source and pure birth was introduced in [1]. The more general cases of a symmetric BRW with finite variance of jumps and one source has been studied by many authors (e.g., [2,3]). Random walks with infinite variance of jumps without branching were widely investigated (see, e.g., [4] and detailed bibliography therein). In [5] it was shown that for BRW the rejection of an assumption about finiteness of variance leads to new phenomena, which is related to the recurrence properties of random walk.

According to the [5], consider the next BRW model. Let $A = (a(x, y))$ for $x, y \in \mathbb{Z}^d$ be the matrix of transition intensities of the random walk, where $a(x, y) \geqslant 0$ for $x \neq y$, $-\infty < a(x, x) < 0$. For the symmetry and spatially homogeneity, $a(x, y) = a(y, x) = a(0, y - x) = a(y - x)$, and $a(0) = -\sum_{z\neq 0} a(z)$. It is assumed that the function $a(\cdot)$ is irreducible, i.e., for each $z \in \mathbb{Z}^d$, there exists a collection of vectors $z_1, \dots, z_k$ such that $z = \sum_{i=1}^{k} z_i$ and $a(z_i) \neq 0$ for $i = 1, \dots, k$.

Let $|\cdot|$ be the Euclidean norm $\mathbb{R}^d$. Suppose now that for all $z \in \mathbb{Z}^d$ with sufficiently large norm, the asymptotic relation

$$a(z) \sim \frac{H\left(\frac{z}{|z|}\right)}{|z|^{(d+\alpha)}},\ \alpha \in (0, 2), \tag{1}$$

holds, where $H(\cdot)$ is a continuous positive function symmetric on the sphere $\mathbb{S}^{d-1} = \{z \in \mathbb{R}^d : |z| = 1\}$, and $a(0) := -\sum_{z\neq 0} a(z)$. This condition leads to the divergence of the series $\sum_{z\neq 0} |z|^2 a(z)$, i.e. to the infinity of the variance of jumps and .

By $p(t, x, y)$ denote the transition probability of the random walk, that at time $t \geqslant 0$ the particle will be at site y, if at time $t = 0$ it is at site x, and determined by the transition intensities $a(x, y)$ (see, e.g., [7]).

The analysis of such models depends on behavior of the probability $p(t, x, y)$ and the Green function, whose properties are closely related to the spectral properties of the operator $\mathcal{A}$ (see [2], [6], [8]). Due to the fact that these functions can be expressed in terms of the Fourier transform of transition intensities, its studying is one of the key points of the analysis of the BRW.

The present work is mainly devoted to the implementation of the Fourier transform studying approach proposed in [9] for $\mathbb{Z}$, $\mathbb{Z}^2$, by using the assertion about multidimensional series with constant coefficients, to $\mathbb{Z}^3$ for the BRW model with infinite variance under assumption (1).

2. Main section

Consider the Fourier transform of the transition intensities $a(\cdot)$ of the described above random walk

$$\phi(\theta) := \sum_{z\in\mathbb{Z}^3} a(z) e^{i\langle z,\theta\rangle}, \quad \theta \in [-\pi, \pi]^d.$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^d$.

From, e.g., [5, 6],

$$p(t, 0, z) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \cos\langle z, \theta\rangle e^{\phi(\theta)t} \, d\theta. \tag{2}$$

As shown in [6], in case of finite variance of jumps the function $\phi(\theta)$ is correctly defined, takes real values, twice continuously differentiable, $\phi(\theta) \leqslant -\gamma|\theta|^2$ for some $\gamma > 0$, and by Theorem 4.1 ch.2 [10] for every $d \geqslant 1$ and every $x, y \in \mathbb{Z}^d$ have $p(t, x, y) \sim \gamma_d t^{-\frac{d}{2}}$ as $t \to \infty$, where $\gamma_d = ((2\pi)^d D_d)^{-\frac{1}{2}}$, $D_d = |\det \phi''(\theta)|$. However, in case of infinite variance of jumps, $\phi''(\theta)$ doesn't exist and the same theorem is not applicable. To obtain the asymptotics of $p(t, x, y)$ it is needed to study the asymptotic behavior of $\phi(\theta)$ and then of the corresponding multidimensional Laplace's integral in (2).

The first problem began to be investigated in [5] for $\mathbb{Z}$ and in [9] for $\mathbb{Z}^2$. The main technical result of the last paper was to estimate the growth rate of $\phi(\theta)$. Then the assertions about multidimensional series with constant coefficients were used. Some generalizations of such series for $d \geqslant 1$ were established in [11], which made it possible to obtain the asymptotics of $\phi(\theta)$. The second problem was solved with the analogue of Watson lemma (see [12]), and the asymptotics for $p(t, x, y)$ of symmetric random walk on $\mathbb{Z}^d$ with infinite variance of jumps was found. It behaves as $h_{\alpha,d} t^{-\frac{d}{\alpha}}$, where $h_{\alpha,d} > 0$ is not depending on x and y for any fixed α, d.

Consider the described in Introduction BRW model with infinite variance of jumps under assumption (1) on $\mathbb{Z}^3$ and apply approach to study the Fourier transform from [9]. The main result is

Theorem 1 *If an function $a(z)$ satisfies listed above assumptions, then the following estimate for its Fourier transform $\phi(\theta)$ is valid:*

$$C|\theta|^\alpha \le |\phi(\theta)|,$$

where $\theta \in [-\pi, \pi]^3$ and $C > 0$.

Note that $\phi(\theta)$ is symmetric due to $a(z) = a(-z)$ model assumption and $H_0 \cdot 3^{-(3+\alpha)} f(\theta) \le |\phi(\theta)| \le H^0 f(\theta)$, where $0 < H_0 \leqslant H(\cdot) \leqslant H^0$, and

$$f(\theta) = \sum_{z \in \mathbb{Z}^3, z \neq 0} \frac{1}{||z||^{3+\alpha}} (1 - e^{i\langle z, \theta \rangle}) = \sum_{k \geqslant 1} \sum_{z \in S_k} \frac{1}{k^{3+\alpha}} (1 - e^{i\langle z, \theta \rangle}).$$

Let $C_k := \{z \in \mathbb{Z}^3 | \|z\| \leqslant k\}$ is the lattice cube containing $(2k+1)^3$ points with center at $0 \in \mathbb{Z}^3$. Besides, $S_k = C_k \setminus C_{k-1}$ and summation on C_k is simpler then on S_k, because in the last case we need to fix consequentially by each two coordinates and vary the third from $-k$ to k, but on C_k we can vary all coordinates without fixing. Therefore, it is expediently to express $f(\theta)$ through C_k.

After transformations for $\sum_{z \in C_k} e^{i\langle z, \theta \rangle} - \sum_{w \in C_{k-1}} e^{i\langle w, \theta \rangle}$ and with designation $S_{3,\alpha}(\varphi) := \sum\limits_{k \geqslant 1} \frac{1}{k^{3+\alpha}} \cos(k\varphi)$ we get

$$f(\theta) = -1/2(C_1(\theta) S_{3,\alpha}(\theta_1 - \theta_2 - \theta_3) + C_2(\theta) S_{3,\alpha}(\theta_1 - \theta_2 + \theta_3) +$$
$$+ C_3(\theta) S_{3,\alpha}(\theta_1 + \theta_2 - \theta_3) + (-C_4(\theta) + 2) S_{3,\alpha}(\theta_1 + \theta_2 + \theta_3)) +$$
$$+ 24 \sum_{k \geqslant 1} \frac{1}{k^{1+\alpha}} + 2 \sum_{k \geqslant 1} \frac{1}{k^{3+\alpha}},$$

where
$C_j(\theta) := (-1)^{\delta_{3j}} \operatorname{ctg}(\theta_1/2) \operatorname{ctg}(\theta_2/2) + (-1)^{\delta_{2j}} \operatorname{ctg}(\theta_1/2) \operatorname{ctg}(\theta_3/2) +$
$+ (-1)^{\delta_{1j}} \operatorname{ctg}(\theta_2/2) \operatorname{ctg}(\theta_3/2) + 1.$

Put $S_\alpha(\varphi) := \sum_{n\geqslant 1}\frac{1}{n^\alpha}\sin(n\varphi)$, $S_{1,\alpha}(\varphi) := \sum_{n\geqslant 1}\frac{1}{n^{1+\alpha}}(1-\cos(n\varphi))$, $S_{2,\alpha}(\varphi) := \sum_{n\geqslant 1}\frac{1}{n^{2+\alpha}}\sin(n\varphi)$. As shown in [9], for $\alpha\in(0,2)$ and $\varphi\to 0+$ the following asymptotic equalities are valid:

$$S_{1,\alpha}(\varphi)\sim \tfrac{1}{\alpha}\Gamma(1-\alpha)\cos\left(\tfrac{1}{2}\pi\alpha\right)\varphi^\alpha,$$
$$S_{2,\alpha}(\varphi)\sim \varphi\sum_{n\geqslant 1}\tfrac{1}{n^{1+\alpha}}-\tfrac{1}{\alpha(1+\alpha)}\Gamma(1-\alpha)\cos\left(\tfrac{1}{2}\pi\alpha\right)\varphi^{\alpha+1}.$$

Note, $S_{3,\alpha}(0)=\sum_{n\geqslant 1}\frac{1}{n^{3+\alpha}}$, $S'_{3,\alpha}(\varphi)=-S_{2,\alpha}(\varphi)$. By integrating the last equality over $[0,\varphi]$ we conclude, as $\varphi\to 0+$

$$S_{3,\alpha}(\varphi)\sim\ L_\alpha-\frac{1}{2}N_\alpha\varphi^2+M_\alpha|\varphi|^{2+\alpha},$$

where $L_\alpha=\sum_{k\geqslant 1}\frac{1}{k^{3+\alpha}}$, $N_\alpha=\sum_{k\geqslant 1}\frac{1}{k^{1+\alpha}}$, $M_\alpha=\frac{\Gamma(1-\alpha)\cos\left(\frac{1}{2}\pi\alpha\right)}{\alpha(1+\alpha)(2+\alpha)}$.

Then for $\theta\in[0,\pi]^3$, $\theta_1\geqslant\theta_2\geqslant\theta_3$, as $|\theta|\to 0$

$$f(\theta)\sim M_\alpha\cdot F_M(\theta)+N_\alpha\cdot F_N(\theta),$$

where functions $F_M(\theta), F_N(\theta)$ have the following complex representation

$$\begin{aligned}F_M(\theta)=&-1/2\cdot(C_1(\theta)\cdot|\theta_1-\theta_2-\theta_3|^{2+\alpha}+C_2(\theta)\cdot|\theta_1-\theta_2+\theta_3|^{2+\alpha}+\\&+C_3(\theta)\cdot|\theta_1+\theta_2-\theta_3|^{2+\alpha}+C_4(\theta)\cdot|\theta_1+\theta_2+\theta_3|^{2+\alpha}),\\F_N(\theta)=&-2\cdot\theta_1\theta_2\operatorname{ctg}(\theta_1/2)\operatorname{ctg}(\theta_2/2)-2\cdot\theta_1\theta_3\operatorname{ctg}(\theta_1/2)\operatorname{ctg}(\theta_3/2)-\\&-2\cdot\theta_2\theta_3\operatorname{ctg}(\theta_2/2)\operatorname{ctg}(\theta_3/2)+\theta_1^2+\theta_2^2+\theta_3^2+24.\end{aligned}$$

For $\theta\in[0,\pi]^3$, $\theta_1\geqslant\theta_2\geqslant\theta_3$, as $|\theta|\to 0$ the next estimations are valid

$$F_M(\theta)\gtrsim C^*|\theta|^\alpha, F_N(\theta)\gtrsim C^{**}|\theta|^2,$$

with $C^*>0$, $C^{**}>0$. For proof it was the convenient way to use the function $\mu(x):=|x|^{2+\alpha}=(x^2)^{\frac{2+\alpha}{2}}$ with $\mu''(x)=(2+\alpha)(1+\alpha)|x|^\alpha$ for $F_M(\theta)$ estimate and to use the expansion $x\cdot\operatorname{ctg}x=1-\frac{1}{2}x^2+o(x^2)$ as $x\to 0$ for $F_N(\theta)$ estimate.

Consequently for $\theta\in[0,\pi]^3$, $\theta_1\geqslant\theta_2\geqslant\theta_3$, as $|\theta|\to 0$ the following inequality holds

$$f(\theta)\gtrsim C'|\theta|^\alpha,$$

where constant $C'>0$. Thus the result of Theorem 1 is obtained.

Hence, we can establish the criteria for transient behavior on $\mathbb{Z}^3$ by analogy with the Theorem 3 in [9]: let $X=(X_t)_{t\geqslant 0}$ be a random walk on $\mathbb{Z}^d$, as it was determined in Introduction, with infinite variance of jumps under assumption (1). Then the random walk X is transient (i.e. $G_0 < \infty$) for $d = 1$ and any $0 < \alpha < 1$, and for $d \geqslant 2$ and any $1 \leqslant \alpha < 2$.

The proof is based on the next relations

$$\int\limits_0^\infty p(t,0,0)dt = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{d\theta}{-\phi(\theta)} \leqslant \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{d\theta}{C|\theta|^\alpha},$$

where $C > 0$, and convergence of the latter integral for $\alpha < d$.

3. Conclusions

The estimation of the Fourier transform for transition intensities of random walk on $\mathbb{Z}^3$ with infinite variance of jumps under assumption (1) by using the assertions about multidimensional series with constant coefficients is established. The criteria for transient behavior of the corresponding random walk based on the mentioned above estimation is considered.

Acknowledgments

The research was supported by the RFBR, project no. 17-01-00468-a.

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