UDC 531.5,519.218

Numerical Solving of Relativistic Schrödinger Equation with Random Quasipotential

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Abstract. In this paper we consider the numerical solutions of the relativistic Schrödinger equation (the singularly perturbed differential equation of the infinite order) with the random quasipotential. We formulate for this equation the boundary value problems on the interval $[0, r_0]$, the boundary value problem with periodic and antiperiodic boundary conditions on the interval $[0, r_0]$ and the boundary value problem on the positive half-line $[0, +\infty)$. The numerical analysis of eigenfunctions and eigenvalues for these boundary value problems is made using MatLab eigenvalues and eigenvectors packages.

Keywords: stochastic process, random quasipotential, relativistic Schrödinger equation, computational methods.

1. Introduction

In the papers [1, 2, 4, 5], the authors considered the relativistic Schröedinger difference equation (Logunov-Tavkhelidze-Kadyshevsky equation, LTK equation) with the quasipotential in the relativistic configurational space for the radial wave functions of bound states for two identical elementary particles without spin

$$\left[H_0^{rad} + V(r) - 2c\sqrt{q^2 + m^2c^2}\right]\psi(r, l) = 0, \tag{1}$$

$$H_0^{rad} = 2mc^2\, ch\left(\frac{i\hbar}{mc}D\right) + \frac{\hbar^2 l(l+1)}{mr(r + \frac{i\hbar}{mc})}\, exp\left(\frac{i\hbar}{mc}D\right)$$

where m is a mass, q is a momentum, l is an angular momentum of each elementary particle and $V(r)$ is a quasipotential (a piecewise continuous function). Asymptotic solutions of boundary value problems for LTK equation was studied in [1,2]. Boundary value problems with the quasipotential were studied on an interval and on a positive half-line for this equation. Using asymptotic methods solutions were obtained in the form of regular and boundary layer parts. Asymptotic behavior of solutions was investigated when a small parameter $\varepsilon \to 0$. In papers [1,2], for LTK equation was made the transition from the infinite order equation to the equation of finite order $2m$. For this "cutting" equation (Logunov-Tavkhelidze-Kadyshevsky truncation equation, LTKT equation) have also been formulated boundary value problems on an interval and on a positive half-line, asymptotic solutions were built for these problems and studied their behavior when the order LTKT equation $m \to \infty$.

In paper [3], solutions of elliptic and parabolic equations was obtained with oscillatory random potentials.

In this paper the numerical analysis of LTK and LTKT equations are investigated with the random quasipotential. It is studying the boundary value problems on the interval $\mathcal{A} \in [0, r_0]$, the boundary value problem with periodic and antiperiodic boundary conditions on the interval $\mathcal{A} \in [0, r_0]$ and the boundary value problem on the positive half-line $\mathcal{B} \in [0, +\infty)$ for obtaining eigenfunctions and eigenvalues.

2. Statement of the boundary value problems

If in the equation (1) we assume that $\hbar = 1, m = 1,\ l = 0$ (S-wave function),

$$\varepsilon = \frac{1}{c}, \lambda_{\varepsilon,\infty} = 2q^2/\sqrt{1+\varepsilon^2 q^2} + 1, q^2 = (1 + 0.25\varepsilon^2\lambda_{\varepsilon,\infty})\lambda_{\varepsilon,\infty},$$

then the equation (1) has the form

$$[\widetilde{L}^{\varepsilon}_{\infty} - \lambda_{\varepsilon,\infty}]\psi_{\varepsilon,\infty}(r) = 0, \tag{2}$$

$$\widetilde{L}^{\varepsilon}_{\infty} = L_2 + \varepsilon^2 L^{\varepsilon}_{\infty} = \sum_{p=1}^{\infty} \varepsilon^{2p-2}\hat{L}_{2p} + V_{\epsilon}(r,\omega),$$

$$\hat{L}_{2p} = \frac{2(-1)^p}{(2p)!!}D^{2p},\ L_2 == -D^2 + V_0(r),\ D^p = d^p/dr^p,$$

where $V_{\epsilon}(r;\omega)$ is a random quasipotential which depends on a small parameter ϵ and $(r;\omega) \in \mathcal{X} \times \Omega$ ($(\Omega; \boldsymbol{F}; \mathcal{P})$ is an abstract probability space for $\mathcal{X} \in \mathcal{A}$ or $\mathcal{X} \in \mathcal{B}$). We suppose that the quasipotential $V_0(r)$ has the deterministic limit $V_{\epsilon}(r;\omega)$ when $\epsilon \to 0$. This differential equation (2) of infinite order with a small parameter ($\varepsilon << 1$) belongs to the class of singularly perturbed equations.

For this differential equation we can formulate the boundary value problem $A^{\infty}_{IBC,\varepsilon}$ on the interval $[0, r_0]$, the boundary value problem $B^{\infty}_{IPBC,\varepsilon}$ with periodic boundary conditions on the interval $[0, r_0]$, the boundary value problem $B^{\infty}_{IAPBC,\varepsilon}$ with antiperiodic boundary conditions on the interval $[0, r_0]$ and the boundary value problem $C^{\infty}_{HLBC,\varepsilon}$ on the positive half-line $[0, +\infty)$ for finding the eigenfunctions $[\psi_{\varepsilon,\infty,\gamma}]^{\infty}_{\gamma=1}$ and eigenvalues $[\lambda_{\varepsilon,\infty,\gamma}]^{\infty}_{\gamma=1}$

$$[\widetilde{L}_{\infty} - \lambda_{\varepsilon,\infty}]\psi_{\varepsilon,\infty}(r) = 0,$$

where

$$D^i\psi_{\varepsilon,\infty}(0) = D^i\psi_{\varepsilon,\infty}(r_0) = 0, \quad i = 0, 1, \ldots,$$

are the boundary conditions for the problem $A^{\infty}_{IBC,\varepsilon}$,

$$D^i\psi_{\varepsilon,\infty}(0) = D^i\psi_{\varepsilon,\infty}(r_0), \quad i = 0, 1, \ldots$$

are the boundary conditions for the problem $B^{\infty}_{IPBC,\varepsilon}$,

$$D^i\psi_{\varepsilon,\infty}(0) = -D^i\psi_{\varepsilon,\infty}(r_0), \quad i = 0, 1, \ldots$$

are the boundary conditions for the problem $B^{\infty}_{IAPBC,\varepsilon}$,

$$D^i\psi_{\varepsilon,\infty}(0) = D^i\psi_{\varepsilon,\infty}(+\infty) = 0, \quad i = 0, 1, \ldots$$

are the boundary conditions for the problem $C^{\infty}_{HLBC,\varepsilon}$.

If we admit that $\varepsilon = 0$ we can formulate degenerate problems for obtaining the eigenfunctions $[\psi_{0,\gamma}]^{\infty}_{\gamma=1}$ and eigenvalues $[\lambda_{0,\gamma}]^{\infty}_{\gamma=1}$

$$[L_2 - \lambda_0]\psi_0(r) = 0,$$

$$\psi_0(0) = \psi_0(r_0) = 0$$

are the boundary conditions for the problem $A_{IBC,0}$,

$$D^i\psi_0(0) = D^i\psi_0(r_0) \quad i = 0, 1,$$

are the periodic boundary conditions for the problem $B_{IPBC,0}$,

$$D^i\psi_0(0) = -D^i\psi_0(r_0) \quad i = 0, 1,$$

are the periodic boundary conditions for the problem $B_{IAPBC,0}$,

$$\psi_0(0) = \psi_0(+\infty) = 0 \tag{3}$$

are the boundary conditions for the problem $C_{HLBC,0}$.

If we admit that the differential equation (2) has a finite order $m > 1$

$$[\widetilde{L}^{\varepsilon}_{2m} - \lambda_{\varepsilon,2m}]\psi_{\varepsilon,2m}(r) = 0,$$

$$\widetilde{L}^{\varepsilon}_{2m} = L_2 + \varepsilon^2 L^{\varepsilon}_{2m} = \sum_{p=1}^{m} \varepsilon^{2p-2}\hat{L}_{2p} + v(r),$$

$$L^{\varepsilon}_{2m} = \sum_{p=1}^{m-1} \varepsilon^{2p-2}\hat{L}_{2p+2} = \sum_{p=1}^{m-1} \frac{2(-1)^{p+1}}{(2p+2)!!}\varepsilon^{2p-2}D^{2p+2},$$

we can formulate the truncation boundary value problem $A^{2m}_{IBC,\varepsilon}$ on the interval $[0, r_0]$, the boundary value problem $B^{2m}_{IPBC,\varepsilon}$ with periodic boundary conditions on the interval $[0, r_0]$, the boundary value problem $B^{2m}_{IAPBC,\varepsilon}$ with antiperiodic boundary conditions on the interval $[0, r_0]$ and the boundary value problem $C^{2m}_{HLBC,\varepsilon}$ on the positive half-line $[0, +\infty)$ for obtaining the eigenfunctions $[\psi_{\varepsilon,2m,\gamma}]_{\gamma=1}^{\infty}$ and eigenvalues $[\lambda_{\varepsilon,2m,\gamma}]_{\gamma=1}^{\infty}$

$$[\widetilde{L}_{2m} - \lambda_{\varepsilon,2m}]\psi_{\varepsilon,2m}(r) = 0, \tag{4}$$

$$D^i\psi_{\varepsilon,2m}(0) = D^i\psi_{\varepsilon,2m}(r_0) = 0, \quad i = 0, 1, \ldots, m-1$$

are the boundary conditions for the truncation problem $A^{2m}_{IBC,\varepsilon}$,

$$D^i\psi_{\varepsilon,2m}(0) = D^i\psi_{\varepsilon,2m}(r_0), \quad i = 0, 1, \ldots, 2m-1$$

are the periodic boundary conditions for the truncation problem $B^{2m}_{IPBC,\varepsilon}$,

$$D^i\psi_{\varepsilon,2m}(0) = -D^i\psi_{\varepsilon,2m}(r_0), \quad i = 0, 1, \ldots, 2m-1$$

are the periodic boundary conditions for the truncation problem $B^{2m}_{IAPBC,\varepsilon}$,

$$D^i\psi_{\varepsilon,2m}(0) = D^i\psi_{\varepsilon,2m}(+\infty) = 0, \quad i = 0, 1, \ldots, m-1$$

are the boundary conditions for the truncation problem $C^{2m}_{HLBC,\varepsilon}$.

If we admit that $\varepsilon = 0$ in (4) we can formulate degenerate problems $A_{IBC,0}$, $B_{IPBC,0}$, $B_{IAPBC,0}$, $C_{HLBC,0}$ for obtaining the eigenfunctions $[\psi_{0,\gamma}]_{\gamma=1}^{\infty}$ and eigenvalues $[\lambda_{0,\gamma}]_{\gamma=1}^{\infty}$.

3. Numerical analysis of the boundary value problems

The numerical analysis of the boundary value problems $A^{\infty}_{IBC,\varepsilon}$ and $A^{2m}_{IBC,\varepsilon}$ on the interval $[0, 1]$, the boundary value problems $B^{\infty}_{IPBC,\varepsilon}$ and $B^{2m}_{IPBC,\varepsilon}$ with periodic boundary conditions on the interval $[0, 1]$, the boundary value problems $B^{\infty}_{IAPBC,\varepsilon}$ and $B^{2m}_{IAPBC,\varepsilon}$ with antiperiodic boundary conditions on the interval $[0, 1]$ and the boundary value problems $C^{\infty}_{HLBC,\varepsilon}$ and $C^{2m}_{HLBC,\varepsilon}$ on the interval $[0, 100]$ and the degenerate problems $A_{IBC,0}$, $B_{IPBC,0}$, $B_{IAPBC,0}$, $C_{HLBC,0}$ for obtaining the eigenfunctions $[\psi_{\varepsilon,\infty,\gamma}]$, $[\psi_{\varepsilon,2m,\gamma}]$, $[\psi_{0,\gamma}]$ and eigenvalues $[\lambda_{\varepsilon,\infty,\gamma}]$, $[\lambda_{\varepsilon,2m,\gamma}]$, $[\lambda_{0,\gamma}]$ is made using MatLab eigenvalues and eigenvectors packages.

The numerical studies were made under the assumption that the random quasipotential $V_\epsilon(r;\omega)$ has the form

$$V_\epsilon(r;\omega) = r^2 + \varepsilon\eta_r,\ V_0(r) = r^2$$

where parameters η_r is an independent normally distributed $N(0, \sigma^2)$ random variable (in the numerical example $\sigma = 0.1$) and $V_0(r)$ is a oscillator quasipotential. The behavior of the eigenfunctions $[\psi_{\varepsilon,2m,\gamma}]$ and eigenvalues $[\lambda_{\varepsilon,2m,\gamma}]$ were studied for cases when $\varepsilon \to 0$ and/or $m \to 0$.

4. Conclusions

In this paper the numerical analysis of the relativistic Schrödinger equation (the singularly perturbed differential equation of the infinite order) with the random quasipotential is made using MatLab eigenvalues and eigenvectors packages. The boundary value problems on the interval $[0, r_0]$, the boundary value problem with periodic and antiperiodic boundary conditions on the interval $[0, r_0]$ and the boundary value problem on the positive half-line $[0, +\infty)$ for obtaining the eigenfunctions and eigenvalues are studied. It is shown the existence of the convergence of the solutions to the deterministic limit for cases when $\varepsilon \to 0$ and/or $m \to 0$.

Acknowledgments

The reported study was funded within the Agreement No 02.a03.21.0008 dated 24.11.2016 between the Ministry of Education and Science of the Russian Federation and RUDN

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