On mean-field $GI/GI/1$ queueing model: existence and uniqueness

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In memory and to the 90th anniversary of

Professor Alexander D. Soloviev

Starting Thanks

1. Thank you the organisers for having invited me. This gives me a great opportunity to pay tribute to my teacher Professor Alexander Dmitrievich Soloviev who offered me several topics to my UG and PG projects; one of them was later published and much later became a reason for me to revisit this area. Yet he never insisted that my topics should be linked to his own research, allowing a great amount of freedom. This talk is sincerely devoted to the memory of Alexander Dmitrievich Soloviev.

2. Thanks to the Russian Academic Excellence Project '5-100' via National Research University Higher School of Economics, Russian Federation, and to the RFBR grant 17-01-00633_a for funding this research.

What is "mean–field" & why important

(λ_{xy} notation will be replaced by λ for jumps up and h for jumps down)

Mean-field version of any stochastic model at an intuitive level presents an ideal "limiting" situation of many similar models with a certain interaction between them ("weak" interaction). It was suggested by A. Vlasov in 1938 to replace the action of all other participants – called sometimes "agents" – by a unique "distributed mean action". Hence, in this limiting model the distribution of the system is involved into the coefficients – which are intensities in our case – of the model itself:

$$\lambda_{xy} = \lambda[x, y, \mu],$$

where μ is a distribution of the model at given time. Most often it is assumed that

$$\lambda[x, y, \mu] = \int \lambda(x, y, z)\, \mu(dz)$$

for a function $\lambda(\cdot)$ which suits the idea of the method.

Mean-field as a limit
(at a physical level)

The "mean-field" reasoning is as follows. Assume there are $N \gg 1$ identical models – in our case $GI/GI/1$ – with more involved intensities λ_{xy} which for any particular device, say, number 1, may depend on the states of each other device as

$$\lambda_{xy} = \frac{1}{N}\sum_{i=2}^{N} \lambda(t, X_t^1, X_t^i, y) \equiv \frac{1}{N}\sum_{i=2}^{N} \lambda[t, X_t^1, \delta_{X_t^i}, y].$$

Due to a "weak interaction" because of the factor $1/N$, the influence of each other device is small. Hence, intuitively they are all weakly denendent and we may reasonably assume a law of large numbers, that is, a weak convergence

$$\frac{1}{N}\sum_{i=2}^{N} \lambda[t, X_t^1, \delta_{X_t^i}, y] \Longrightarrow \int \lambda(t, X_t^1, z, y)\, \mu_t(dz),$$

where $\mu_t(dz)$ is the (limiting at $N \to \infty$) distribution of each device at time t. This is exactly the special case of what we denoted by $\lambda[t, X_t^1, \mu_t]$ & $h[t, X_t^1, \mu_t]$ on the previous slide.

Model; state space; markovization; intensities

(Assume that i.v. X_0 does have $x^1 \geq 0$ the elapsed time from the last arrival)

The "markovian" state space of the process (X_t) is the union

$$\mathcal{X} := (0, x^1) \cup \bigcup_{n=1}^{\infty} (n, x^1, x^2), \quad x^1, x^2 \geq 0.$$

Here n is the number of "customers" in the system; the value x^1 stands for the elapsed *time from the last arrival*, while x^2 signifies the elapsed *time of the current service*. There is only one server which works without breaks (if there is at least one customer in the system) and it is always in a working state. All newly arrived customers stand in a queue of the infinite capacity, and for simplicity only we assume the FIFO discipline of service. It is assumed that at any time t at any state $X = (n, x^1, y^2)$ (or $X = (0, x^1)$ for $n = 0$) there are *intensities* of service $h[t, X_t, \mu_t]$ and arrivals $\lambda[t, X_t, \mu_t]$, where μ_t is the distribution of the random variabe X_t itself.

Terminology

Queueing processes with general (non-exponential) intensities or service & arrival distribution functions are often called **piecewise-linear Markov processes**. They are not Markov in the natural state space $\mathbb{Z}_+$ but are Markov at special moments of time. The term was introduced by Gnedenko & Kovalenko. Later on M.H.A. Davis generalized it to **piecewise-deterministic Markov processes**. They are all **semi-Markov**. Of course, they all can be made Markov via an extension of a state space, which is used here via accepting the state space $\mathcal{X}$ instead of $\mathbb{Z}_+$.

On the other hand, mean-field models are often called **nonlinear Markov**.

So, our object to be constructed in this talk might be called **nonlinear piecewise-linear Markov process**, or, **piecewise-linear nonlinear Markov process**.

Model & assumptions: *denote* $x = (x^1, x^2)$, $y = (y^1, y^2)$
λ is intensity of arrivals, h intensity of service; NB: t, x^1, x^2 all signify times

The model is a standard (FIFO) $GI/GI/1/\infty$ except that both intensities are more involved. Note the difference in notations $\lambda(\ldots)$ & $\lambda[\ldots]$: the real *intensity* is $\lambda[\ldots]$.

(A1)

$$\lambda[t, x, \mu] = \int \lambda(t, x, y)\mu(dy),$$

$$h[t, x, \mu] = \int h(t, x, y)\mu(dy).$$

(A2) $\lambda(t, x, y)$ and $h(t, x, y)$ are Borel and bounded.

(A3) $\lambda(t, x, y)$ and $h(t, x, y)$ are continuous in all variables.

(A4) $\lambda(t, x, y)$ and $h(t, x, y)$ are bounded away from zero. *(That is, $\lambda \wedge h \geq c > 0$ uniformly in all variables.)*

Results: Theorem 1, Existence

(A1-A3): $(\lambda, h)[t, x, \mu] = \int (\lambda, h)(t, x, y)\mu(dy)$ bounded and continuous

For $X = (n, x^1, x^2) \in \mathcal{X}$ and $\hat{X} = (\hat{n}, \hat{x}^1, \hat{x}^2) \in \mathcal{X}$ denote

$$X^+ := (n+1, 0, x^2), \;\; X^- := 1(n > 0)(n-1, x^1, 0),$$

$$L[t, \hat{X}, \mu]g(X) := \lambda[t, \hat{X}, \mu](g(X^+) - g(X)) + \frac{\partial}{\partial x^1} g(n, x^1, x^2)$$

$$+ 1(n > 0)h[t, \hat{X}, \mu](g(X^-) - g(X)) + 1(n > 0)\frac{\partial}{\partial x^2} g(n, x^1, x^2).$$

Theorem (1)

Under (A1)–(A3), for any fixed $X_0 \in \mathcal{X}$, on some probability space there exists a Markov process $(X_t, \; t \geq 0)$ with generator L, and marginal distributions μ_t, and intensities $\lambda[t, X_t, \mu_t], \; h[t, X_t, \mu_t]$.

Here boundedness away from zero – (A4) – is not required.

Theorem 1: in other words

($\exists$ a Markov process (X_t) with intensities $\lambda[t, X_t, \mu_t]$, $h[t, X_t, \mu_t]$)

In other words, there exists a Markov process (X_t) such that for any bounded continuous function $g(X)$ with bounded continuous derivatives in (x, y), the expression

$$M_t := g(X_t) - g(X_0) - \int_0^t L[s, X_s, \mu_s]g(X_s)\,ds \tag{1}$$

is a martingale, where (reminder) for $X = (n, x, y)$, $X' = (n', x', y')$, $n \geq 0$, $t \geq 0$, it is denoted $X^+ := (n+1, 0, y), X^- := 1(n > 0)(n-1, x, 0)$,

$$\begin{aligned} L[t, X', \mu]g(X) &:= \lambda[t, X', \mu](g(X^+) - g(X)) \\ &+ 1(n > 0)h[t, X', \mu](g(X^-) - g(X)) \\ &+ \frac{\partial}{\partial x}g(n, x, y) + 1(n > 0)\frac{\partial}{\partial y}g(n, x, y). \end{aligned} \tag{2}$$

Theorem 1: moreover

($\exists$ a Markov process $(X_t,\ t \geq 0)$ with intensities $\lambda[t, X_t, \mu_t],\ h[t, X_t, \mu_t]$)

Moreover, for any given measure-valued function $(\mu_s,\ s \geq 0)$ in $L[s, X_s, \mu_s]$, the *martingale problem* (see, e.g., [Ethier–Kurtz]) (3) has a weakly unique solution (= unique in distribution in the space of trajectories).

Moreover, for any *locally* bounded continuous function $g(X)$ with *locally* bounded continuous derivatives in (x, y), the expression

$$M_t := g(X_t) - g(X_0) - \int_0^t L[s, X_s, \mu_s] g(X_s)\, ds \qquad (3)$$

is a *local* martingale (i.e., a martingale if stopped by any suitable *localizing sequence* of stopping times).

Theorem 1: yet one more reformulation

((local) mart $M_t := g(X_t) - g(X_0) - \int_0^t L[s, X_s, \mu_s] g(X_s)\, ds$)

Equivalently, *Dynkin's indentity* holds true for any function $g(X)$ from the same class as for a (non-local) mart,

$$\mathbb{E}_{0,X_0} g(X_t) = g(X_0) + \mathbb{E}_{0,X_0} \int_0^t L[s, X_s, \mu_s] g(X_s)\, ds. \qquad (4)$$

Moreover, equivalently, for any $0 \le t_1 < t_2 \ldots < t_{m+1}$, and for any Borel bounded functions $\phi_k(X)$, $X \in \mathcal{X}$,

$$\mathbb{E}_{0,X_0} \left(g(X_{t_{m+1}}) - g(X_{t_m}) - \int\limits_{t_m}^{t_{m+1}} L[s, X_s, \mu_s] g(X_s)\, ds \right) \times \prod_{k=1}^{m} \phi_k(X_{t_k}) = 0. \qquad (5)$$

All Dynkin's formulae may be understood as a *complete probability* – or, rather, *a complete expectation* – rule.

Sketch of the proof

Skorokhod's lemma

Lemma (Studies in the theory of random processes, Ch.1.6)

Let ξ^n_t ($t \geq 0$, $n = 0, 1, \ldots$) be some d-dimensional stochastic processes defined on some probability space and let for any $T > 0$, $\epsilon > 0$ the following hold true:

$$\lim_{c\to\infty} \sup_n \sup_{t\leq T} \mathbb{P}(|\xi^n_t| > c) = 0,$$

$$\lim_{h\downarrow 0} \sup_n \sup_{t,s\leq T;\,|t-s|\leq h} \mathbb{P}(|\xi^n_t - \xi^n_s| > \epsilon) = 0.$$

Then there exists a subsequence $n' \to \infty$ and a new probability can be constructed with processes $\tilde{\xi}^{n'}_t$, $t \geq 0$ and $\tilde{\xi}_t$, $t \geq 0$, such that all finite-dimensional distributions of $\tilde{\xi}^{n'}_\cdot$ coincide with those of $\xi^{n'}_\cdot$ and such that for any $\epsilon > 0$ and all $t \geq 0$,

$$\mathbb{P}(|\tilde{\xi}^{n'}_t - \tilde{\xi}_t| > \epsilon) \to 0, \quad n' \to \infty.$$

Skorokhod lemma's conditions

how to verify? $|X| = |(n, x, y)| := |n| + |x| + |y|$

They easily follow under the asssumption (A2) – boundedness:

(1)

$$\mathbb{P}(\sup_{0\le t\le T} |X_t^n| > c) \overset{\text{Chebyshev–Markov}}{\le} \mathbb{P}(|X_0^n| > c/2)$$

$$+\frac{\mathbb{E}\sup_{0\le t\le T} |X_t^n - X_0^n|}{c} \le \mathbb{P}(|X_0| > c/2) + \frac{\|\lambda\| T}{c/2};$$

(2)

$$\mathbb{P}(|X_t^n - X_s^n| > \epsilon) \le 1 - \exp(-(\|\lambda\| + \|h\|)|t-s|)$$
$$\le (\|\lambda\| + \|h\|)|t-s|.$$

I: Existence. This part can be based on Picard iterations and Skorokhod's lemma. For any $n \geq 1$ consider a process (X_t^n), with initial data $X_0^n = X_0$ and intensities of jumps up and down, respectively,

$$\lambda[t, X^n_{[nt]/n}, \mu^n_{[nt]/n}], \quad h[t, X^n_{[nt]/n}, \mu^n_{[nt]/n}].$$

where $[a]$ is the integer value for $a \in \mathbb{R}$ and where X_t^n with $t < 0$ is understood as X_0^n, and similarly for μ_t^n. The processes (X_t^n) for each n are constructed by induction successfully on the intervals $[0, 1/n], [1/n, 2/n]$, etc. Due to the boundedness assumption on both intensities, there is no blow up and the processes for any n are defined for any $t \geq 0$ as càdlàg pure jump processes. Moreover, for any t probability of jump at t for any X^n equals zero.

(Features of the approximating process (X^n_t))

$\lambda[t, X^n_{[nt]/n}, \mu^n_{[nt]/n}], \quad h[t, X^n_{[nt]/n}, \mu^n_{[nt]/n}]; \kappa_n(a) := [na]/n$

Firstly, a bad news is that (X^n) is regrettfully not Markov in the normal state space. A good news is that it behaves in a way very similar to an MP and here we briefly discuss one its important property: the analogue of Dynkin's formula,

$$\mathbb{E}_{0,X_0}\left[\left(g(X^n_{t_{m+1}}) - g(X^n_{t_m}) - \int\limits_{t_m}^{t_{m+1}} \mathbb{E}' L(s, X^n_{\kappa_n(s)}, \xi^n_{\kappa_n(s)}) g(X^n_s)\, ds\right) \prod_{k=1}^{m} \phi_k(X^n_{t_k})\right] = 0,$$

for any (nonrandom) $0 \le t_1 < \ldots < t_m < t_{m+1}$ and for any g bounded and continuous and with bounded and continuous derivatives with respect to (x, y) (recall that $X = (n, x, y)$ and $\kappa_n(a) := [na]/n$).

(Features of the approximating process, ctd)
equivalent statements; recall that $\kappa_n(a) := [na]/n$

$$\mathbb{E}_{0,X_0}\Bigg[\Bigg(g(X^n_{t_{m+1}}) - g(X^n_{t_m}) - \int_{t_m}^{t_{m+1}} \mathbb{E}' L(s, X^n_{\kappa_n(s)}, \xi^n_{\kappa_n(s)}) g(X^n_s)\, ds\Bigg) \prod_{k=1}^m \phi_k(X^n_{t_k})\Bigg] = 0,$$

$$\mathbb{E}_{0,X_0}\Bigg[\Bigg(g(X^n_{t_{m+1}}) - g(X^n_{t_m}) - \int_{t_m}^{t_{m+1}} L(s, X^n_{\kappa_n(s)}, \mu^n_{\kappa_n(s)}) g(X^n_s)\, ds\Bigg) \prod_{k=1}^m \phi_k(X^n_{t_k})\Bigg] = 0 =$$

$$= \mathbb{E}_{0,X_0}\Big((g(X^n_{t_{m+1}}) - g(X^n_{t_m}) - \int_{t_m}^{t_{m+1}} L(s, X^n_{\kappa_n(s)}, \mu^n_{\kappa_n(s)}) g(X^n_s)\, ds) | \mathcal{F}_{t_m}\Big).$$

Mean-field
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The last formula is close to "normal" Dynkin's identity for MP and can be understood as a "complete conditional expectation" formula,

$$\mathbb{E}_{0,X_0}((g(X^n_{t_{m+1}}) - g(X^n_{t_m}) - \int_{t_m}^{t_{m+1}} L(s, X^n_{\kappa_n(s)}, \mu^n_{\kappa_n(s)}) g(X^n_s)\, ds)|\mathcal{F}_{t_m}) = 0.$$

Indeed, recall the definition of L:

$$X^+ := (n+1, 0, y),\ \ X^- := 1(n>0)(n-1, x, 0),$$

$$L[t, X', \mu]g(X) := \lambda[t, X', \mu](g(X^+) - g(X)) + \frac{\partial}{\partial x} g(n,x,y)$$

$$+ 1(n>0) h[t, X', \mu](g(X^-) - g(X)) + 1(n>0)\frac{\partial}{\partial y} g(n,x,y).$$

Hence, the "quasi-Dynkin's" formula can be rewritten as

(“Complete conditional expectations”, rewritten)

(distinguish between n and n_s!) the formula can be rewritten as follows:

$$
\begin{aligned}
&\mathbb{E}_{0,X_0}((g(X^n_{t_{m+1}})|\mathcal{F}_{t_m}) \stackrel{\text{a.s.}}{=} g(X^n_{t_m}) \\
&+ \mathbb{E}_{0,X_0} \int_{t_m}^{t_{m+1}} \lambda[s, X^n_{\kappa_n(s)}, \mu_{\kappa_n(s)}](g(X^+_s) - g(X_s))\, ds|\mathcal{F}_{t_m}) \\
&+ \mathbb{E}_{0,X_0} \int_{t_m}^{t_{m+1}} 1(n_s > 0)h[t, X^n_{\kappa_n(s)}, \mu_{\kappa_n(s)}](g(X^-_s) - g(X_s))\, ds|\mathcal{F}_{t_m}) \\
&+ \mathbb{E}_{0,X_0} \int_{t_m}^{t_{m+1}} (\frac{\partial}{\partial x} g(X^n_s) + 1(n_s > 0)\frac{\partial}{\partial y} g(X^n_s))\, ds)|\mathcal{F}_{t_m}).
\end{aligned}
$$

By inspection, it has all features of the conditional complete expectation: λds is a infinitesimal proba of jump up, etc.

(The plan of a rigorous justification)
for the "quasi-Dynkin's formula"

This "complete conditional expectation" can be rigorously justified in a similar way as in [V-v, Zverkina, MPRF, 2014] where it was done in a nonconditional markovian setting. This way consists of rewriting the integrals as approximating sums *(because the probability as well as the integral are sigma-additive but generally no more – i.e. not "uncountably-additive"; at least, if we want to use such uncountable version of additivity then it must be somehow rigorously justified, which is apparently an exceptional event in such an applied area as queueing theory),* then of taking into account infinitesimal probabilities of jumps up and down, estimating all $o(\Delta)$ where $\Delta \downarrow 0$, and finally passing to the limit so as to get integrals again.

(Existence)

(Return to our approximating procedure (X^n) with $\lambda[t, X^n_{\kappa_n(t)}, \mu^n_{\kappa_n(t)}]$, etc.)

The processes $(X_t^n,\ t \geq 0)$ for $n \geq 1$ being constructed, let us introduce on some probability space *independent* equivalent processes $(\xi_t^n,\ t \geq 0)$; let $\mathbb{E}'$ stand in all cases for the integration with respect to the *third variable*, e.g.,

$$\mathbb{E}' h(t, X_t^n, \xi_t^n) := \int h(t, X_t^n, y) \mu_t^n(dy).$$

It can be checked that due to boundedness of both intensities (in fact, here we, of course, have a potential to relax our conditions) the assumptions of Skorokhod's Lemma are satisfied: for any $T > 0$, $\epsilon > 0$,
$\lim_{c\to\infty} \sup_n \sup_{t\leq T} \mathbb{P}(|\tilde{X}_t^n| > c) = 0$, &
$\lim_{h\downarrow 0} \sup_n \sup_{t,s\leq T;\, |t-s|\leq h} \mathbb{P}(|\tilde{X}_t^n - \tilde{X}_s^n| > \epsilon) = 0$

(Existence)

Hence, on some new probability space there are equivalent – and, hence, *satisfying similar "quasi-Dynkin's identities" with the same* L^1 – processes $(\tilde X^n_t, \tilde\xi^n_t)$ and a limiting pair $(\tilde X_t, \tilde\xi_t)$ such that for some subsequence $(\tilde X^{n'}_t, \tilde\xi^{n'}_t) \stackrel{\mathbb{P}}{\to} (\tilde X_t, \tilde\xi_t)$, $n' \to \infty$, for each t. It follows due to the boundedness of all intensities that the limiting process $(\tilde X_t, \tilde\xi_t)$ is also stochastically continuous. More than that, it is a pure jump process with a finite number of jumps on any bounded interval with probability one. Moreover, due to $\lim_{h\downarrow 0} \sup_n \sup_{t,s\le T;\, |t-s|\le h} \mathbb{P}(|\tilde X^n_t - \tilde X^n_s| > \epsilon) = 0$ for any $\epsilon > 0$ it follows that (uniformly with respect to $0 \le t \le T$)

$$\tilde X^{n'}_{\kappa_{n'}(t)} \stackrel{\mathbb{P}}{\to} \tilde X_t, \quad n' \to \infty.$$

[1] "Quasi" just because the processes are not Markov and L is not a generator; but very close to it intuitively.

(5) $\mathbb{E}_{0,X_0}\left(g(X_{t_{m+1}}) - g(X_{t_m}) - \int_{t_m}^{t_{m+1}} L[s, X_s, \mu_s] g(X_s)\, ds\right) \prod_{k=1}^m \phi_k(X_{t_k}) = 0$

So, our “quasi-Dynkin's” formula being valid,

$$\mathbb{E}_{0,X_0}\Bigg[\Bigg(g(\tilde{X}^{n'}_{t_{m+1}}) - g(\tilde{X}^{n'}_{t_m}) - \int_{t_m}^{t_{m+1}} \mathbb{E}' L(s, \tilde{X}^{n'}_{\kappa_{n'}(s)}, \tilde{\xi}^{n'}_{\kappa_{n'}(s)}) g(\tilde{X}^{n'}_s)\, ds\Bigg) \prod_{k=1}^m \phi_k(\tilde{X}^{n'}_{t_k})\Bigg] = 0, \tag{6}$$

as $n' \to \infty$, in the limit due to the convergence & C_b continuity assumption (A3) on λ, h, we get $\forall\, \phi_k \in C_b$,

$$\mathbb{E}_{0,X_0}\left(g(\tilde{X}_{t_{m+1}}) - g(\tilde{X}_{t_m}) - \int_{t_m}^{t_{m+1}} \mathbb{E}' L(s, \tilde{X}_s, \tilde{\xi}_s) g(\tilde{X}_s)\, ds\right) \times \prod_{k=1}^m \phi_k(\tilde{X}_{t_k}) = 0.$$

(Existence)

mart = martingale

Since the law $\tilde\mu_t$ of $\tilde\xi_t$ is the same as of $\tilde X_t$ then equivalently

$$\mathbb{E}_{0,X_0}\left(g(\tilde X_{t_{m+1}}) - g(\tilde X_{t_m}) - \int_{t_m}^{t_{m+1}} Lg(s,\tilde X_s,\tilde\mu_s)\,ds\right) \times \prod_{k=1}^{m}\phi_k(\tilde X_{t_k}) = 0. \qquad (7)$$

By the properties of measures on $\mathbb{R}^d$, the formula (7) holds true for any Borel bounded (ϕ_k), too. Due to [Davis], solution of the "martingale problem" (7) – or, more precisely, of the mart-problem

$$M_t := g(\tilde X_{t_{m+1}}) - g(\tilde X_{t_m}) - \int_{t_m}^{t_{m+1}} Lg(s,\tilde X_s,\tilde\mu_s)\,ds \quad \textit{is a mart},$$

with given $(\tilde\mu_s)$ is unique. By Krylov's result on "selection of a MP" (cf. [Ethier-Kurtz, ∼Thm 4.4.2]) $\tilde X$ is Markov with a generator L.

[Krylov; Ethier-Kurtz – Theorem 4.4.2]
reference

In an abstract form, the re-phrased result by [Krylov 1973] as in [Ethier-Kurtz, Theorem 4.4.2 & around] claims that *if the martingale problems has a unique solution in the sense of distributions then the corresponding process is Markov*. Actually [Ethier-Kurtz] established that *just uniqueness of marginal distributions suffices for this*.

The fact that L serves as a generator follows directly from the definition after passing to the limit.

Note that uniqueness in the *mean-field setting* was not proved yet! Uniqueness in the theorem above is stated for a fixed flow of one-dimensional distributions $(\tilde{\mu}_t)$ in the intensities.

Results: Theorem 2, Uniqueness

(A1-A2,A4): $(\lambda, h)[t, x, \mu] = \int (\lambda, h)(t, x, y)\mu(dy)$ bdd and strictly positive

A quick reminder about the assumptions:

(A2) The functions $\lambda(t, x, y)$ and $h(t, x, y)$ are Borel measurable and bounded.

(A4) The functions $\lambda(t, x, y)$ and $h(t, x, y)$ are bounded away from zero.

Theorem (2)

Let the assumptions (A1)–(A2) and (A4) be satisfied. Then, for any fixed X_0, the process $(X_t,\ t \geq 0)$ with required intensities $\lambda[t, x, \mu_t]$ and $h[t, x, \mu_t]$ is unique in distribution.

NB: continuity (A3) is not required here!

Skorokhod-Girsanov's formula

Proof is based on the "Skorokhod-Girsanov formula for jump processes".

Suppose there are two solutions, (X^1_t, μ^1_t) and (X^2_t, μ^2_t), $t \geq 0$. The density on $[0, T]$ – see, e.g., [Liptser-Shiryaev, Theory of Martingales], or [Skorokhod, Studies ...] – of one measure w.r.t. another reads,

$$\rho_T := \frac{d\mathbb{P}^{(\lambda,h)[\cdot,X,\mu^2]}}{d\mathbb{P}^{(\lambda,h)[\cdot,X,\mu^1]}}(X)|_{\mathcal{F}_T} \quad \text{("Skorokhod-Girsanov")}$$

$$= \prod_{i=1}^{n} \frac{\lambda[t_i, X_{t_i}, \mu^2_{t_i}]}{\lambda[t_i, X_{t_i}, \mu^1_{t_i}]} \exp\left(-\int_0^T (\lambda[t, X_t, \mu^2_t] - \lambda[t, X_t, \mu^1_t])\, dt\right)$$

$$\times \prod_{j=1}^{m} \frac{h[s_j, X_{s_j}, \mu^2_{s_j}]}{h[s_j, X_{s_j}, \mu^1_{s_j}]} \exp\left(-\int_0^T (h[t, X_t, \mu^2_t] - h[t, X_t, \mu^1_t])\, dt\right),$$

where (t_i) are the moments of jumps up and (s_j) – for jumps down of the trajectory X; the usual $\prod_{i=1}^{0} \ldots = 1$ is assumed.
Note that due to (A4) the ratios λ/λ & h/h are bounded.

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Indeed, consider a generic event $A = A_{n,m}$: n jumps up and m down occur in a $B_u \cup B_d$ (up & down) with $B_u \cap B_d = \emptyset$. We have with $\lambda^k_{t_i} = \lambda[t_i, X_{t_i}, \mu^k_{t_i}]$, $h^k_{s_j} = h[s_j, X_{s_j}, \mu^k_{s_j}]$, $k = 1, 2$,

$$\mathbb{P}^{\lambda[\cdot,X,\mu^1]}(A) = \int_{B_u} \exp\left(-\int_0^T \lambda[t, X_t, \mu^1_t])\, dt\right) \prod_{i=1}^n \lambda^1_{t_i}\, dt_i$$

$$\times \int_{B_d} \exp\left(-\int_0^T h[t, X_t, \mu^1_t])\, dt\right) \prod_{j=1}^m h^1_{s_j}\, ds_j,$$

$$\mathbb{P}^{\lambda[\cdot,X,\mu^2]}(A) = \int_{B_u}\int_{B_d} \rho_T \exp\left(-\int_0^T \lambda[t, X_t, \mu^1_t])\, dt\right) \prod_{i=1}^n \lambda^1_{t_i}\, dt_i$$

$$\times \exp\left(-\int_0^T h[t, X_t, \mu^1_t])\, dt\right) \prod_{j=1}^m h^1_{s_j}\, ds_j.$$

We want to estimate the distance *(note a bit less usual definition!)* $\|\mu^1_{[0,T]}-\mu^2_{[0,T]}\|_{TV}=:\sup_A(\mu^1_{[0,T]}(A)-\mu^2_{[0,T]}(A))$ between two probability measures in the space of trajectories, and then to use the inequality

$$\varphi_T:=\|\mu^1_T-\mu^2_T\|_{TV}\le\|\mu^1_{[0,T]}-\mu^2_{[0,T]}\|_{TV}=1-\mathbb{E}^{\lambda^1}(\rho_T\wedge 1)=:\psi_T.$$

We need a suitable *lower* bound for the value $\mathbb{E}^{\lambda^1}(\rho_T\wedge 1)$. Let us split this expectation as follows:

$$\mathbb{E}^{\lambda^1}(\rho_T\wedge 1)=\sum_{n,m=0}^{\infty}\mathbb{E}^{\lambda^1}1(A_{n,m})\,(\rho_T\wedge 1)\,,$$

where – recall – $A_{n,m}$ is the event that the trajecotry X has precisely n jumps up & m down on $[0,T]$.

(Uniqueness: the basis for the main calculus)

A fact: $\int f(y)\mu^1(dy) - \int f(y)\mu^2(dy) \le 2\|f\|_B \|\mu^1 - \mu^2\|_{TV}$

Denote $\Lambda^k_t = \ln \lambda^k[t, X_t, \mu^k_t]$, $H^k_t = \ln h^k[t, X_t, \mu^k_t]$. Note that $\|\Lambda\| + \|H\| \le C < \infty$ with some $0 < C < \infty$ under (A4). We have for $n, m = 0$ a lower bound,

$$\mathbb{E}^{\lambda^1} 1(A_{0,0}) \, (\rho_T \wedge 1) = \mathbb{E}^{\lambda^1} 1(A_{0,0})$$

$$\times \exp\left(- \int_0^T ((\lambda + h)[t, X_t, \mu^2_t] - (\lambda + h)[t, X_t, \mu^1_t]) \, dt \right) \wedge 1$$

$$\ge \exp(-(\|\lambda\| + \|h\|) \, 2T\psi_T) \mathbb{E}^{\lambda^1} 1(A_{0,0}).$$

Now we want to find similar bounds for all other $A_{n,m}$ and then – remember that eventually we are estimating $1 - \ldots$; to start with, we have $1 - \exp(-CT) \le CT$ where the most important is the multiplier T. *(Color just emphasizes the importance.)*

(Uniqueness: the basis for the main calculus)
(ctd)

Similarly to $A_{0,0}$, in the general case for $n, m \geq 0$,

$$\mathbb{E}^{\lambda^1} 1(A_{n,m}) \left(\rho_T \wedge 1\right)$$

$$= \mathbb{E}^{\lambda^1} 1(A_{n,m}) \left\{ \prod_{i=1}^{n} \frac{\lambda[t_i, X_{t_i}, \mu^2_{t_i}]}{\lambda[t_i, X_{t_i}, \mu^1_{t_i}]} \prod_{j=1}^{m} \frac{h[s_j, X_{s_j}, \mu^2_{s_j}]}{h[s_j, X_{s_j}, \mu^1_{s_j}]} \times \right.$$

$$\left. \times \exp\left(-\int_0^T ((\lambda + h)[t, X_t, \mu^2_t] - (\lambda + h)[t, X_t, \mu^1_t])\, dt \right) \right\} \wedge 1$$

$$\geq \ldots$$

(Uniqueness: the basis for the main calculus ctd.))
(by definition $t_0 = 0$)

$$
\begin{aligned}
&\geq \mathbb{E}^{\lambda^1} \int \cdots \int \exp(-\sum_{i=1}^{n} 2\|\Lambda\| \, \|\mu^2_{t_i} - \mu^1_{t_i}\|_{TV}) \\
&\times \exp(-\sum_{j=1}^{m} 2\|H\| \, \|\mu^2_{s_j} - \mu^1_{s_j}\|_{TV}) \\
&\times \exp(-\int_0^T 2(\|\lambda\| + \|h\|)\|\mu^2_t - \mu^1_t\|_{TV} \, dt) \\
&\times \prod_{i=1}^{n} \lambda[t_i, X_{t_i}, \mu^1_{t_i}] \exp(-\int_{t_{i-1}}^{t_i} \lambda[s, X_s, \mu^1_s] \, ds) \, dt_i \\
&\times \prod_{j=1}^{m} h[s_j, X_{s_j}, \mu^1_{s_j}] \exp(-\int_{s_{j-1}}^{s_j} h[s, X_s, \mu^1_s] \, ds) \, ds_j.
\end{aligned}
$$

(Uniqueness) $1 = \sum_{n,m=0}^{\infty} 1(A_{n,m})$, $1 - \exp(-a) \le a$

$\varphi_T = \|\mu_T^1 - \mu_T^2\|_{TV} \le \|\mu_{[0,T]}^1 - \mu_{[0,T]}^2\|_{TV} = 1 - \mathbb{E}^{\lambda^1}(\rho_T \wedge 1) = \psi_T$

So, $$\psi_T \le (1 - \exp(-2(\|\lambda\| + \|h\|)\, T\psi_T))\mathbb{E}^{\lambda^1} 1(A_{0,0})$$

$$+ \sum_{|(n,m)|>0}^{\infty} \mathbb{E}^{\lambda^1} 1(A_{n,m}) \left(1 - \exp\left(-2\sum_{i=1}^{n} \|\Lambda\|\, \|\mu_{t_i}^2 - \mu_{t_i}^1\|_{TV} - \right.\right.$$

$$\left.\left. -2\sum_{j=1}^{m} \|H\|\, \|\mu_{s_j}^2 - \mu_{s_j}^1\|_{TV}\right)\right) \exp\left(-\int_0^T 2(\|\lambda\| + \|h\|)\|\mu_t^2 - \mu_t^1\|_{TV}\, dt\right)$$

$$\le \sum_{n,m=0}^{\infty} \mathbb{E}^{\lambda^1} 1(A_{n,m})(1 - \exp(-2(n\|\Lambda\| + m\|H\|)\psi_T - T(\|\lambda\| + \|h\|)\psi_T))$$

$$\le \psi_T \left(\sum_{n,m=0}^{\infty} (2n\|\Lambda\| + 2m\|H\| + T(\|\lambda\| + \|h\|)) \left\{ \frac{((\|\lambda\| + \|h\|)T)^{n+m}}{(n+m)!} \right\} \right)$$

(Uniqueness)

$\varphi_T = \|\mu^1_T - \mu^2_T\|_{TV} \le \|\mu^1_{[0,T]} - \mu^2_{[0,T]}\|_{TV} = 1 - \mathbb{E}^{\lambda^1}(\rho_T \wedge 1) = \psi_T$

Overall,

$$\psi_T \le \psi_T \times$$

$$\times \left(\sum_{n+m=0}^{\infty} ((n+m)(\|\lambda\| \vee \|h\|) + T(\|\lambda\| + \|h\|)) \frac{(\|(\lambda\| + \|h\|)T)^{n+m}}{(n+m)!}\right)$$

$$\stackrel{m+n=:k}{=} T\,\psi_T\,\{(\|\lambda\| + \|h\|)$$

$$+ \sum_{k=1}^{\infty} (k(\|\Lambda\| + \|H\|) + T(\|\lambda\| + \|h\|))\,\frac{(\|\lambda\| + \|h\|)^k T^{k-1}}{k!}\Big\}.$$

This series converges and does not exceed some constant, say, $C > 0$, if $T \le 1$. So, $\boxed{(0 \le \psi_T \le TC\psi_T, \quad T \le 1}$, which implies that $\psi_T = 0, \ T < C^{-1} \wedge 1$, and, therefore, also

$$\varphi_T = 0, \quad T < C^{-1} \wedge 1,$$

as required.

(Uniqueness)

$\varphi_T = \|\mu^1_T - \mu^2_T\|_{TV} \le \|\mu^1_{[0,T]} - \mu^2_{[0,T]}\|_{TV} = 1 - \mathbb{E}^{\lambda^1}(\rho_T \wedge 1) = \psi_T$

In other words, we have shown that the two marginal measures μ^1_t and μ^2_t coincide for all $t < C^{-1} \wedge 1$.

Further, note the constant C in this calculus does not depend on the initial distribution of the process. Hence, using the Markov property of the process and repeating the same arguments on $[T, 2T]$, $[2T, 3T]$, etc., by induction we conclude that

$$\psi_t = 0, \quad t \ge 0,$$

and, therefore, also

$$\varphi_t = 0, \quad t \ge 0,$$

as required. So, the two measures μ^1 and μ^2 on the space of trajectories are equal. The Theorem 2 is proved.

Further problems?
which study did not start yet

1. Try drop the assumptions (A3) & (A4)(?)
2. A further natural problem is to investigate whether there is a weak convergence towards our model of the sequence of models with an increasing number of interacting "normal" (non-mean-field) GI/GI/1 sub-models all with identical and independent initial states where interactions are in the intensities of jumps written for the first model as an example

$$\frac{1}{N-1}\sum_{j=2}^{N}\lambda(t,X_t^1,X_t^j) \equiv \frac{1}{N-1}\sum_{j=2}^{N}\lambda[t,X_t^1,\delta_{X_t^j}]\ (\Longrightarrow ?)$$

Intuitively at the "physical level" each element in this ensemble weakly affects one another, so that a law of large numbers *may* occur as $N \to \infty$, which is called "propagation of chaos". This LLN, if valid, should lead exactly to our mean-field model.
3. To study more general jump/jump-diffusion mean-filed models. For no-jump SDEs see [Mishura, V-v, 2017, arXiv:1603.02212v4]

Final Thanks

It is my hope that *Professor Soloviev* would have been delighted to see new developments and – less important – perhaps also his former student returning to the area of queueing after years in stochastic calculus including some mean-field versions of stochastic differential equations, and I *am grateful to him for teaching me* at the university and for drawing my attention to important classes of problems which interest me until now and are still far from being solved.

And, of course, may I thank you all for having come to this talk and for your attention.